Romanian Academy
Institute of Mathematics "Simion Stoilow"

Doctoral Thesis

– summary –

Applications of duality in some inifinite dimensional optimization problems

Ph.D. Advisor:
CS I dr. Dan Tiba

Ph.D. Student
Diana-Rodica Merluşcă

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Introduction

Variational inequalities are a subject of large interest in mathematics, physics or informatics, due to the numerous applications. One of the problems that can be formulated involving variational inequalities is the obstacle problem. There is a direct link between the obstacle problem and the free boundary problems, as Lewy and Stampacchia [90] showed, and their solution reduces frequently to optimization problems with constraints.

The main goal of this work is to present a series of duality based algorithms for the variational problems associated with elliptic equations and inequations. The original results included in Chapter 2 are published in the papers Merluşcă [100], [101] and [103], and those in Chapter 3, in Merluşcă [102]. The used methodology is an extension of the ideas introduced by Sprekels and Tiba [128], Neittaanmaki, Sprekels and Tiba [107].

Key words: obstacle problem, Fenchel Theorem, approximate problem, approximate methods, biharmonic operator.

1 Mathematical background

In this chapter we summarize some mathematical notions and results regarding functional analysis, Sobolev spaces, optimization problems, duality theory, variational equations and inequalities and approximation methods.

2 Second order problems

We apply a duality based method to the second order general obstacle problem and show that its approximate solving reduces to finding the solution of a finite dimensional quadratic minimization problem. In the mathematical literature, there are other duality approaches, different from the ones introduced here. Ito and Kunisch [79] introduced a primal-dual active set strategy and proved that it is equivalent to the semi-smooth Newton method. An approach using Fenchel's duality theorem and the semi-smooth Newton method was used, in Hintermüller and Rösel [78], for obtaining some results involving semi-static contact problems.

2.1 The duality type method for null obstacle problems

We discuss the obstacle problem in the Sobolev spaces $W_0^{1,p}(\Omega)$, with $p > dim\ \Omega$. The main idea is to solve the problem using an approximate one and its dual. We apply Fenchel's theorem to analyse the obtained dual problem. We show that the solution of the dual approximate problem is, in fact, a linear combination of Dirac distributions. In conclusion, solving a quadratic minimization problem we can build the approximate solution of the primal problem by simply applying a formula which relates the primal and dual solutions.

Consider $\Omega \subset \mathbb{R}^n$ a bounded domain with the strong local Lipschitz property. We study the obstacle problem

$$\min_{y \in W_0^{1,p}(\Omega)_+} \left\{ \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \right\} \tag{1}$$

where $f \in L^1(\Omega)$, $p > n = dim\ \Omega$ and $W_0^{1,p}(\Omega)_+ = \{y \in W_0^{1,p}(\Omega) : y \geq 0 \text{ in } \Omega\}$.

By the Sobolev imbedding theorem. we have $W^{1,p}(\Omega) \to C(\overline{\Omega})$ and it makes sense to consider the approximate problem

$$\min \left\{ \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \ \ : \ y \in W_0^{1,p}(\Omega); y(x_i) \geq 0, i = 1, 2, \dots, k \right\} \tag{2}$$

where $\{x_i\}_{i \in \mathbb{N}} \subseteq \Omega$ is a dense set in Ω. For all $k \in \mathbb{N}$, we denote the closed convex cone $C_k = \{y \in W_0^{1,p}(\Omega) : y(x_i) \geq 0, i = 1, 2, \dots, k\}$.

Proposition 2.1. *Problem* (1) *has a unique solution* $\bar{y} \in W_0^{1,p}(\Omega)_+$ *and for all* $k \in \mathbb{N}$ *problem* (2) *has a unique solution* $\bar{y}_k \in C_k$.

Moreover, we obtain the following result

Theorem 2.2. *The sequence* $\{\bar{y}_k\}_k$ *of the solutions of problems* (2), *for* $k \in \mathbb{N}$, *is a strongly convergent sequence in* $W^{1,p}(\Omega)$ *to the unique solution* $\bar{y}$ *of the problem* (1).

We apply Fenchel's duality theorem to obtain the dual problems associated to the problems (1) or (2). To this end, we consider the functional

$$F(y) = \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy, \quad y \in W_0^{1,p}(\Omega). \tag{3}$$

Let q be the exponent conjugate to p. Using the definition of the convex conjugate and the fact that the duality mapping $J : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is a single-valued and bijective operator, we get that the convex conjugate of F is $F^*(y^*) = \frac{1}{2}\|f + y^*\|^2_{W^{-1,q}(\Omega)}$

Consider the functional $g_k = -I_{C_k}$. the concave conjugate is

$$g_k^\bullet(y^*) = \inf \{(y, y^*) - g_k(y) : y \in C_k\} = \begin{cases} 0, & y^* \in C_k^* \\ -\infty, & y^* \notin C_k^* \end{cases}$$

where $C_k^* = \{y^* \in W^{-1,q}(\Omega) : (y^*, y) \geq 0, \forall y \in C_k\}$.

Lemma 2.3. *The polar cone of* C_k *is*

$$C_k^* = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{x_i} : \alpha_i \geq 0 \right\}$$

where δ_{x_i} *are the Dirac distributions concentrated at* $x_i \in \Omega$*, i.e.* $\delta_{x_i}(y) = y(x_i)$*,* $\forall y \in W_0^{1,p}(\Omega)$*.*

Since the domain of g_k is $D(g_k) = C_k$ and the functional F is continuous on the closed convex cone C_k, the hypotheses of Fenchel duality Theorem are satisfied. This implies that

$$\min_{y \in C_k} \left\{ \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega f y \right\} = \max_{y^* \in C_k^*} \left\{ -\frac{1}{2} \|y^* + f\|^2_{W^{-1,q}(\Omega)} \right\} \tag{4}$$

So we obtain the dual approximate problem associated to problem (2)

$$\min \left\{ \frac{1}{2} \|y^* + f\|^2_{W^{-1,q}(\Omega)} : y^* \in C_k^* \right\}. \tag{5}$$

Theorem 2.4. *Let* $\bar{y}_k$ *be the solution of the approximate problem* (2) *and* $\bar{y}_k^*$ *the solution of the dual approximative problem* (5)*. Then the two solutions are related by the formula*

$$\bar{y}_k = J^{-1}(\bar{y}_k^* + f) \tag{6}$$

where J *is the duality mapping* $J : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$*. Moreover,* $(\bar{y}_k^*, \bar{y}_k) = 0$*.*

Remark 2.5. Since $\bar{y}_k^* \in C_k^*$, using Lemma 2.3, it yields that $\alpha_i^* \bar{y}_k(x_i) = 0, \quad \forall i = 1, 2, \ldots, k$ In conclusion, the Lagrange multipliers α_i^* are zero if $\bar{y}_k(x_i) > 0$ and they can be positive only when the constraint is active, i.e. $\bar{y}_k(x_i) = 0$.

2.2 The duality-type method for a general obstacle problem

We extend here the duality method to the general obstacle problem. We reduce the problem to the null obstacle case and we compute the solutions using the duality method presented above. We first show that the initial obstacle may be replaced with another one having zero trace on the boundary and the problem has still the same solution. Afterwards, we perform a translation to the null obstacle case and we can apply the theory form the previous section.

We consider the following obstacle problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla y|^2 - \int_\Omega f y \quad : \; y \in K_\psi \right\}, \tag{7}$$

where $K_\psi = \{y \in H_0^1(\Omega) : y \geq \psi\}$, $\psi \in H^1(\Omega)$, $\psi|_{\partial\Omega} \leq 0$ and $f \in L^2(\Omega)$.

The unique solution of problem (7) is an element of $H^2(\Omega)$.

Lemma 2.6. *Let y_ψ be the solution of the problem* (7) *and $\hat{y}$ the solution of the problem*

$$\begin{aligned} -\Delta \hat{y} = f, &\quad \text{on } \Omega\,, \\ \hat{y} = 0, &\quad \text{on } \partial\Omega\,, \end{aligned} \tag{8}$$

then $y_\psi \geq \hat{y}$ almost everywhere on Ω.

The problem (7) *in which we replace ψ by $\hat{\psi} = \max\{\hat{y}, \psi\} \in H_0^1(\Omega)$ has the same solution y_ψ.*

The problem that we obtain after translation is

$$\min_{y \in K_0} \left\{ \frac{1}{2} \int_\Omega |\nabla y|^2 - \int_\Omega f y + \int_\Omega \nabla \hat{\psi} \nabla y \right\} . \tag{9}$$

where $K_0 = \{y \in H_0^1(\Omega) : y \geq 0 \text{ a.p.t. } \Omega\} = (H_0^1(\Omega))^+$.

The problem has again a unique solution, considering that the functional $\int_\Omega (fy - \nabla \hat{\psi} \nabla y)$ is linear. Let y_0 be this solution.

Proposition 2.7. *The solution of the problem* (7) *can be computed by*

$$y_\psi = y_0 + \hat{\psi}\,. \tag{10}$$

To apply the above results, we now impose the condition $p > dim\ \Omega$, that is $dim\ \ \Omega = 1$. We shall work in the familiar Sobolev space $H_0^1(\Omega)$ ($p = 2$).

We define $\hat{f} \in H^{-1}(\Omega)$ as $(\hat{f}, y)_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_\Omega (fy - \nabla \hat{\psi} \nabla y), \quad \forall y \in H_0^1(\Omega)$. Consider the approximate problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla y|^2 - (\hat{f}, y)_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \ : \ y \in C_k \right\} , \tag{11}$$

where $C_k = \{y \in H_0^1(\Omega) : y(x_i) \geq 0, \forall i = 1, 2, \ldots, k\}$ and $\{x_i\}_i$ is a dense set in Ω.

Proposition 2.8. *There exists a unique solution $y_k^0 \in C_k$ of the problem* (11).

Using the Sobolev imbedding theorem and the weak lower semicontinuity of the norm, we can prove the following approximation result

Theorem 2.9. *The sequence $\{\bar{y}_k\}_k$ of the solutions of the problems* (11), *for $k \in \mathbb{N}$, is a strongly convergent sequence in $H_0^1(\Omega)$ to the unique solution $\bar{y}$ of the problem* (9).

Applying the Fenchel duality theorem to problem (11) we obtain the dual problem

$$\min \left\{ \frac{1}{2} |y^* + \hat{f}|^2_{H^{-1}(\Omega)} \ \ : \ y^* \in C_k^* \right\} , \tag{12}$$

where $C_k^* = \{y^* \in H^{-1}(\Omega) : y^* = \sum_{i=1}^k \alpha_i \delta_{x_i}, \alpha_i \geq 0\}$ is the dual cone.

Remark 2.10. Let $\hat{y}_k^*$ be the solution of the dual approximate problem (12). Since $\hat{y}_k^* \in C_k^*$, it is sufficient to compute the coefficients α_i^*. The solution y_k^0 of the approximate problem (11) is computed using the identity $y_k^0 = J^{-1}(\hat{y}_k^* + \hat{f})$ (Theorem 2.4), where J is the duality mapping $J : H_0^1(\Omega) \to H^{-1}(\Omega)$. We also have $\alpha_i^* y_k(x_i) = 0, \quad \forall i = \overline{1,k}$.

We obtain the formula for the solution of the approximate problem, denoted by y_k^0,

$$y_k^0 = \sum_{i=1}^{k} \alpha_i^* J^{-1}(\delta_{x_i}) + J^{-1}(\hat{f})$$

using the fact that the duality mapping $J : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by $J(y) = -y''$. Applying (10) we find the approximate solution of the general obstacle problem (7).

2.3 Numerical applications

In this section we apply the above theoretical results to the obstacle problem for second order operators in dimension one. We comment here just one of the examples discussed in the thesis.

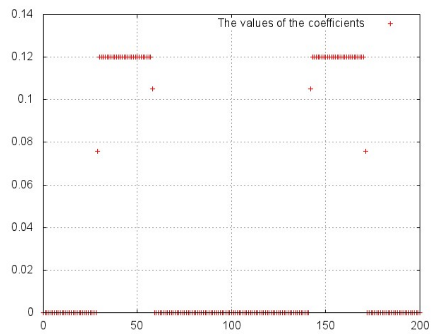

Figure 1: The dual approximate solution.

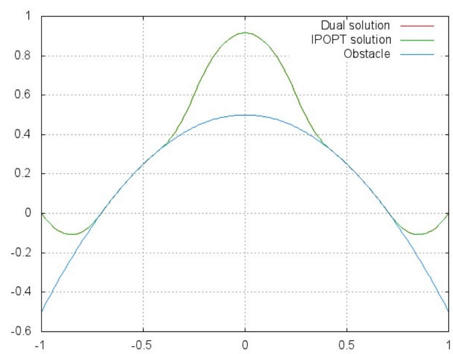

Figure 2: The duality based solution and the `IPOPT` solution.

Example 2.1. We consider the general obstacle problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla y|^2 - \int_\Omega f y \ \ : \ y \in K_\psi \right\}, \tag{13}$$

where $K_\psi = \{y \in H_0^1(\Omega) : y \geq \psi\}$, $\Omega = (-1, 1)$, $\psi(x) = -x^2 + 0.5$ and

$$f(x) = \begin{cases} -10, & |x| > 1/4\,, \\ 10 - x^2, & |x| \leq 1/4\,. \end{cases}$$

We represent in Figure 1 the dual approximate solution and in Figure 2 the obstacle ψ and the solutions, one computed by the duality method and the other one computed by the `IPOPT` method [137]. The two solutions coincide graphically.

3 Fourth order problems

The obstacle problem for the biharmonic operator is an intensely researched subject in mathematics. Among the many works that treat the problem, we cite Caffarelli, Friedman and Torelli [37], An, Li and Li [8], Anedda [10], Landau and Lifshitz [89], Brezis and Stampacchia [33] or Comodi [45].

Many authors have used duality ideas in solving plate related problems. We mention here the work of Yau and Gao [141] that establishes a generalized duality principal, based on a nonlinear version of Rockafellar's duality theory [118] and obtains a dual semi-quadratic problem for the von Kármán obstacle problem. We also recall the works of Neittaanmaki, Sprekels and Tiba [107] and Sprekels and Tiba [128] devoted to the Kirchhoff-Love arches and obtaining explicit formulas for the solution.

3.1 The simply supported plate problem with null obstacle

We consider that $\Omega \subset \mathbb{R}^n$, with $n \leq 3$, a bounded domain with the strong local Lipschitz property. We denote by V the space $H^2(\Omega) \cap H_0^1(\Omega)$ endowed with the scalar product $(u, v)_V = \int_\Omega \Delta u \Delta v$. The norm $|y|_V = \left(\int_\Omega (\Delta y)^2\right)^{\frac{1}{2}}$ is equivalent to the usual Sobolev norm.

Consider the following obstacle problem

$$\min_{y \in K} \left\{ \frac{1}{2} \int_\Omega (\Delta y)^2 - \int_\Omega f y \right\} \tag{14}$$

where $f \in L^2(\Omega)$ and $K = \{y \in V : y \geq 0 \text{ in } \Omega\}$, which is a simplified model of the simply supported plate.

By the Sobolev theorem, and using the fact that $dim\ \Omega \leq 3$, we have $H^2(\Omega) \cap H_0^1(\Omega) \to C(\overline{\Omega})$ and thus we may consider the following approximate problem

$$\min \left\{ \frac{1}{2} \int_\Omega (\Delta y)^2 - \int_\Omega f y \ : \ y \in V; y(x_i) \geq 0, i = 1, 2, \ldots, k \right\} \tag{15}$$

where $\{x_i\}_{i \in \mathbb{N}} \subseteq \Omega$ is a dense set in Ω. For each $k \in \mathbb{N}$, we denote the closed convex cone $C_k = \{y \in V : y(x_i) \geq 0, i = 1, 2, \ldots, k\}$.

Proposition 3.1. *Problem* (14) *has a unique solution* $\bar{y} \in K$ *and problem* (15) *has a unique solution* $\bar{y}_k \in C_k$*, for each* $k \in \mathbb{N}$.

Furthermore, we have the following approximation result

Theorem 3.2. *The sequence* $\{\bar{y}_k\}_k$ *of the solutions of problems* (15) *is a strongly convergent sequence in* V *to the unique solution* $\bar{y}$ *of the problem* (14).

We denote V^* the dual space of V. Notice that $H^{-2}(\Omega)$ is not dense in V^*, since $H_0^2(\Omega)$ is not dense in V. But the inclusion $H_0^2(\Omega) \subset V$ is continuous, then for every $y^* \in V^*$ the restriction $y^*|_{H_0^2(\Omega)} \in H^{-2}(\Omega)$. We obtain the following result

Lemma 3.3. *The duality mapping* $J : V \to V^*$ *is defined by*

$$J(v) = \Delta\Delta v.$$

By Fenchel's duality Theorem, the dual problem associated to (15) is

$$\min\left\{\frac{1}{2}|y^* + f|_{V^*}^2 : y^* \in C_k^*\right\}. \tag{16}$$

where we show that $C_k^* = \left\{u = \sum_{i=1}^k \alpha_i \delta_{x_i} : \alpha_i \geq 0\right\}$ as in Lemma 2.3.

Theorem 3.4. *Let* $\bar{y}_k$ *be the solution of the approximate problem* (15) *and* $\bar{y}_k^*$ *the solution of the dual associated problem* (16). *Then* $\bar{y}_k = J^{-1}(\bar{y}_k^* + f)$ *where* J *is the duality mapping* $J : V \to V^*$.

Moreover, $(\bar{y}_k^*, \bar{y}_k)_{V^* \times V} = 0$.

Remark 3.5. Again we have $\alpha_i^* \bar{y}_k(x_i) = 0, \quad \forall i = 1, 2, \ldots, k$.

3.2 The clamped plate problem

We focus now on the clamped plate with null obstacle. We develop a similar theory as above. The differences emerge from the fact that the maximum principal doesn't hold in general for the boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial n} = 0, \quad \text{pe } \partial\Omega.$$

We again consider $\Omega \subset \mathbb{R}^n$, cu $n \leq 3$ a bounded domain with the strong local Lipschitz property. Here, we denote by V the Hilbert space $H_0^2(\Omega)$ endowed with the scalar product $(u, v)_V = \int_\Omega \Delta u \Delta v$.

The obstacle problem is

$$\min_{y \in \mathcal{K}}\left\{\frac{1}{2}\int_\Omega (\Delta y)^2 - \int_\Omega fy\right\} \tag{17}$$

where $f \in L^2(\Omega)$ and $\mathcal{K} = \{y \in V : y \geq 0 \text{ in } \Omega\}$.

The problem (17) has the unique solution $\bar{y} \in \mathcal{K}$.

By the Sobolev theorem, and using the fact that $dim\ \Omega \leq 3$, we have $H_0^2(\Omega) \to C(\overline{\Omega})$ and it makes sense to consider the following approximate problem

$$\min\left\{\frac{1}{2}\int_\Omega (\Delta y)^2 - \int_\Omega fy \ : \ y \in V; y(x_i) \geq 0, i = 1, 2, \ldots, k\right\} \tag{18}$$

where $\{x_i\}_{i\in\mathbb{N}} \subseteq \Omega$ is a dense set in Ω. For each $k \in \mathbb{N}$, we consider the closed convex cone $\mathcal{C}_k = \{y \in V : y(x_i) \geq 0, i = 1, 2, \dots, k\}$.

For all $k \in \mathbb{N}$ we denote by $\bar{y}_k \in \mathcal{C}_k$ the unique solution of the approximate problem (18).

In this case as well, the following approximate result holds

Theorem 3.6. *The sequence* $\{\bar{y}_k\}_k$ *of the solutions to the problems* (18), *for* $k \in \mathbb{N}$, *is a strongly convergent sequence in* V *to the unique solution* $\bar{y}$ *of the problem* (17).

The dual approximate problem associated with problem (18) is

$$\min\left\{\frac{1}{2}|y^* + f|^2_{V^*} : y^* \in \mathcal{C}_k^*\right\}. \tag{19}$$

For this case we have a similar result as in the case of the simply supported plate.

Theorem 3.7. *Consider* $\bar{y}_k$ *to be the solution of the approximate problem* (18) *and* $\bar{y}_k^*$ *the solution of the dual approximate problem* (19). *Then* $\bar{y}_k = J^{-1}(\bar{y}_k^* + f)$ *where* J *is the duality mapping* $J : V \to V^*$. *Moreover,* $(\bar{y}_k^*, \bar{y}_k) = 0$.

Remark 3.8. As before we notice that the complementarity relation $\alpha_i^* \bar{y}_k(x_i) = 0, \quad \forall i = 1, 2, \dots, k$ still holds.

3.3 Numerical applications and comparison of the dual method with other methods

We apply the algorithms on some examples and compare the results with other numerical methods. We indicate here just the case of simply supported plates, but in the thesis clamped plates are computed as well.

Example 3.1. We take Ω the unit disc in $\mathbb{R}^2$ and we solve the simply supported plate obstacle problem

$$\min_{y\in K}\left\{\frac{1}{2}\int_\Omega (\Delta y)^2 - \int_\Omega fy\right\} \tag{20}$$

where $K = \{y \in H_0^1(\Omega) \cap H^2(\Omega) : y \geq 0 \text{ in } \Omega\}$ and $f(x_1, x_2) = 100(-x_1^2 + 3x_1)$.

We computed two solutions. The one by the dual method is represented in Figure 3 and the one by the IPOPT method [137] is represented in Figure Figura 4 and we notice that they are different.

Comparing the values in Table 1, we conclude that the computed minimum values of the cost functional are lower when applying the dual method. This shows that the duality methods generates a more precise solution.

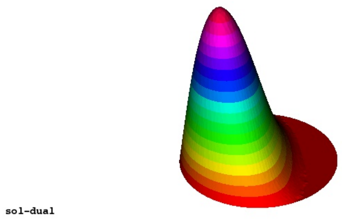

Figure 3: The solution obtained using the duality based method.

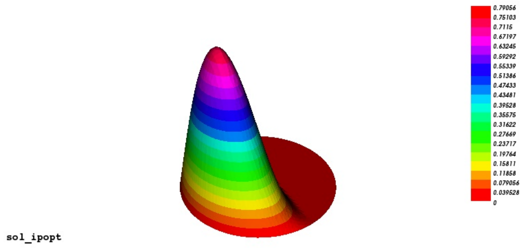

Figure 4: The solution given by the direct method.

Table 1: The values of the energy functional for different meshes with the number of vertices denoted by k.

k	205	682	1031	1431	1912	2797
IPOPT	-55.8069	-57.9099	-58.168	-58.3493	-58.4457	-58.5392
Dual	-78.0675	-80.5279	-80.8705	-81.113	-81.2397	-81.3977

Example 3.2. In the case of fourth order operators, the reduction procedure to null obstacles generalizes the ideas from second order operators. Supplementary difficulties appear due to the loss of the regularity properties.

We consider $\Omega = (0, 2) \times (0, 1)$ and $f(r) = -10(-2r^2 + 20r - 2)$, with $r = \sqrt{x^2 + y^2}$. We take the general obstacle $\psi(r) = -r^2 + 2r - 1.5$.

We represent the solution computed by the duality type method in Figure 5 and in Figure 6 the solution obtained by the `IPOPT` method [137].

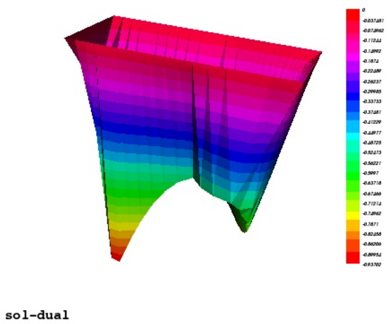

Figure 5: The solution given by the duality based method.

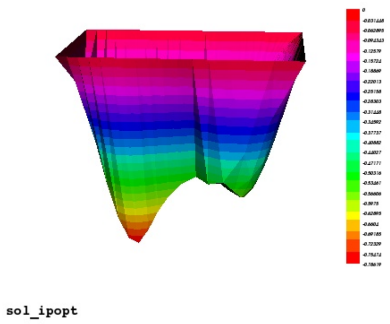

Figure 6: The solution obtained using the direct method.

The two solutions are not identical, but Table 2 shows that the duality based method generates lower optimal values of the energy functional by

Table 2: Optimal values of the energy functional obtained on meshes with various number of vertices denoted by k.

k	322	484	716	1430	1920	2568
IPOPT	-105.675	-108.804	-107.047	-104.101	-103.9	-103.802
Dual	-118.551	-121.568	-121.447	-118.268	-118.135	-118.143

comparison with the case in which the direct `IPOPT` method is used.

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